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S. L. Gavrilyuk[*], M. Yu. Kazakova [†,‡]

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Dedicated to L. V. Ovsyannikov on the occasion of his 95^{th} birthday.

Abstract

A closure relation is proposed for description of hydraulic jumps in two-layer flows with a free surface over a horizontal bottom. Such a relation comes from the momentum equations of each layer which become in a sense conservative when the total momentum and the masses in each layer are conserved. It is also shown that this relation guarantees that the energy flux decreases through the jump.

1 Governing equations

Consider a flow of two immiscible heavy fluid layers of depths h_1, h_2 over a horizontal bottom (see Figure 1). In the long wave approximation the following model can be derived (Ovsyannikov *et al.*, 1985, Baines, 1995):

$$(\gamma_1 h_1)_t + (\gamma_1 h_1 u_1)_x = 0, \tag{1}$$

$$(\gamma_2 h_2)_t + (\gamma_2 h_2 u_2)_x = 0, \tag{2}$$

$$\gamma_1 h_1 (u_{1t} + u_1 u_{1x}) + g h_1 (\gamma_1 h_1 + \gamma_2 h_2)_x = 0, \tag{3}$$

$$\gamma_2 h_2 (u_{2t} + u_2 u_{2x}) + g h_2 \gamma_2 (h_1 + h_2)_x = 0. \tag{4}$$

Here γ_1 , γ_2 ($\gamma_1 > \gamma_2$) are constant densities, and u_1, u_2 are velocities in each layer. The gravity acceleration is denoted by g.

It was recently shown (Montgomery and Moodie, 2001, Barros, 2006) that, in addition to the mass conservation laws, the only conservation laws admitted by the system are the conservation of the total momentum

$$(\gamma_1 h_1 u_1 + \gamma_2 h_2 u_2)_t + (\gamma_1 h_1 u_1^2 + \gamma_2 h_2 u_2^2 + E)_x = 0, \tag{5}$$

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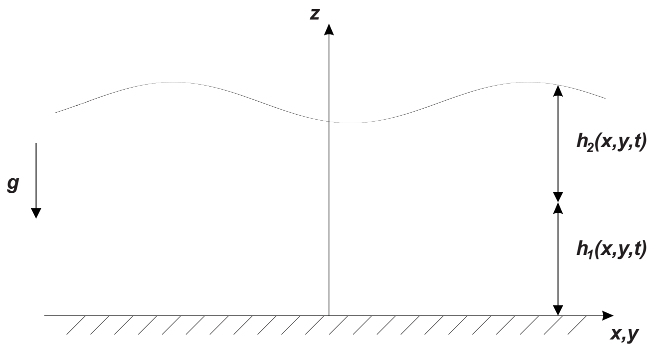

Figure 1: A physical picture of a two-layer flow with a free surface over a flat bottom

the total energy

$$\left(\frac{\gamma_1 h_1 u_1^2}{2} + \frac{\gamma_2 h_2 u_2^2}{2} + E\right)_t \tag{6}$$

$$+\left(u_1 h_1\left(\frac{\gamma_1 u_1^2}{2} + g(\gamma_1 h_1 + \gamma_2 h_2)\right) + u_2 h_2\left(\frac{\gamma_2 u_2^2}{2} + g\gamma_2(h_1+h_2)\right)\right)_x = 0,$$

and the Bernoulli integrals

$$\gamma_1 u_{1t} + \left(\frac{\gamma_1 u_1^2}{2} + g\left(\gamma_1 h_1 + \gamma_2 h_2\right)\right)_x = 0, \tag{7}$$

$$\gamma_2 u_{2t} + \left(\frac{\gamma_2 u_2^2}{2} + g\gamma_2\left(h_1 + h_2\right)\right)_x = 0. \tag{8}$$

Here the "internal energy" E is defined as

$$E = g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2.$$

The corresponding characteristic polynomial can easily be derived from (1) - (4) :

$$\left((u_1-\lambda)^2 - gh_1\right)\left((u_2-\lambda)^2 - gh_2\right) - g^2\frac{\gamma_2}{\gamma_1}h_1 h_2 = 0. \tag{9}$$

The system is conditionally hyperbolic provided the relative velocity $w = u_1 - u_2$ is small (Ovsyannikov, 1979, Baines, 1995, Barros, 2006, Abgrall and Karni, 2009). The hyperbolicity implies a possibility of shock formation (corresponding to hydraulic jumps) even if the initial data are smooth.

A dispersive regularization of (1), (2), (3) and (4) has been derived in Liska, Margolin and Wendroff (1995) and studied in Barros, Gavrilyuk and Teshukov (2007) and Barros and Gavrilyuk (2007). In particular, solitary wave solutions of the dispersive equations have been studied in the last two references. For the case where the velocities at infinity in each layer are equal, it has been found that dispersive equations admit "table-top" (and "table-down") internal solitary waves for some singular values of Froude numbers. The amplitude of the free surface waves is, in general, much smaller than the amplitude of internal waves. The half of such a "table-top" soliton can be considered as a regular "hydraulic jump" where the total momentum and the total energy are conserved.

The singular hydraulic jumps considered for the two − layer shallow water system (1), (2), (3) and (4) are a symplification of ondular bores having nonstationary oscillating tail after the first smooth front representing a solitary wave. Such ondular bores have been intensively studied in one-layer shallow water flows (El, Grimshaw and Smyth, 2006, LeMetayer, Gavrilyuk and Hank, 2010), however their nonstationary analysis is always absent for the two-layer case.

An old question concerning singular hydraulic jumps in two-layer flows is : what are the Rankine-Hugoniot relations determining such a jump? The mass conservation laws (1), (2) and the conservation of the total momentum (5) are obvious candidates. The energy conservation law plays the role of the entropy inequality through the jumps (see the corresponding discussion in the case of one-layer flows in Stoker, 1957). Finally, one of the Bernoulli conservation laws can be used, but which one? The same problem appears in two-layer flows between rigid lids (see Baines, 1995, §3.5) where one of the possible closing hypotheses could be to privilege the Bernoulli equation in the contracting layer. However, as noticed in Baines, 1995, "this assumption can not be strictly correct, as observations show that there is some dissipation occurring in *each* layer". The use of the Bernoulli laws for the two-layer flows with free surface is thus also questionable.

Studying shear flows in homogeneous fluids, Liapidevskii and Teshukov (2000) proposed to use the local vorticity to closure the jump relations. In the case of two-layer flows an analogue of such a vorticity is the velocity difference. Ostapenko (2001) has been proved that if we choose the layer depths h_1, h_2, the total momentum $\gamma_1 h_1 u_1 + \gamma_2 h_2 u_2$ and the velocity difference $u_1 - u_2$ as the basic variables, the domain of convexity of the total energy as a function of such variables is maximal compared with other choices of conservative variables (for example, h_1, h_2, u_1, u_2). He considered it as a mathematical argument justifying the choice of the conservation law for the relative velocity. Also, in the case of very weak stratification ($\gamma_1 \approx \gamma_2$), in Liapidevskii and Teshukov (2000) the choice of the conservation law for the relative velocity has been used to solve the Riemann problem for two-layer flows under a rigid lid. In particular, such a choice allowed them to construct a solution of the Riemann problem in the hyperbolicity region satisfying the Lax stability condition on the hydraulic jumps.

Abgrall and Karni (2009) used the rigid-lid assumption at shocks to study numerically the two-layer flows over topography. The same problem with jump conditions resulting from the classical choice of a linear path through the space state was studied in Mandli (2011). The jump relations in the last two cases are not, *a priori*, compatible with the entropy inequality. However, this inequality was always checked *a posteriori*.

In this paper, we exploit another idea proposed by V. Teshukov in 2007. He remarked that the non-conservative momentum equations for each layer become conservative at the variety where the mass of each layer and the total momentum are conserved. In particular, we will show that such a closure is compatible with the entropy inequality.

2 Basic relations

We suppose the hydraulic jump is stationary in the reference system moving with the jump. The mass and the total momentum equations (1), (2), (5) can then be integrated :

$$\gamma_1 h_1 u_1 = Q_1 = const, \tag{10}$$

$$\gamma_2 h_2 u_2 = Q_2 = const, \tag{11}$$

$$\gamma_1 h_1 u_1^2 + \gamma_2 h_2 u_2^2 + g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2 = M = const. \tag{12}$$

Eliminating the velocities, the momentum equation becomes

$$\frac{Q_1^2}{\gamma_1 h_1} + \frac{Q_2^2}{\gamma_2 h_2} + g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2 = M = const. \tag{13}$$

For given γ_1, γ_2 ($\gamma_1 > \gamma_2$) and small Q_1 and Q_2, the function

$$M(h_1, h_2) = \frac{Q_1^2}{\gamma_1 h_1} + \frac{Q_2^2}{\gamma_2 h_2} + g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2 \tag{14}$$

has only one non-degenerate critical point which is the minimum point. Indeed, consider the function $M(h_1, h_2)$ defined by (14). Its Hessian matrix (denoted by $M''(h_1, h_2)$) is positive definite

$$M''(h_1, h_2) = \begin{pmatrix} \frac{2Q_1^2}{\gamma_1 h_1^3} + g\gamma_1 & g\gamma_2 \\ g\gamma_2 & \frac{2Q_2^2}{\gamma_2 h_2^3} + g\gamma_2 \end{pmatrix} > 0$$

because $\gamma_1 > \gamma_2$. The minimum point of $M(h_1, h_2)$ is defined from the equations :

$$h_1^2 \left(h_1 + \frac{\gamma_2}{\gamma_1} h_2 \right) = \frac{Q_1^2}{g\gamma_1^2}, \tag{15}$$

$$h_2^2 \left(h_2 + h_1 \right) = \frac{Q_2^2}{g\gamma_2^2}.$$

The level curves of $M(h_1, h_2)$ in the vicinity of the critical point are then closed.

Consider the total energy

$$\mathcal{E} = \frac{\gamma_1 h_1 u_1^2}{2} + \frac{\gamma_2 h_2 u_2^2}{2} + g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2.$$

The partial derivatives of the energy with respect to velocities u_i and depths h_i define the fluxes of the Bernoulli integrals (7), (8) and the mass fluxes(1), (2) (Bridges and Donaldson, 2009) :

$$\frac{\partial \mathcal{E}}{\partial h_1} = \frac{\gamma_1 u_1^2}{2} + g(\gamma_1 h_1 + \gamma_2 h_2) = R_1, \qquad \frac{\partial \mathcal{E}}{\partial u_1} = \gamma_1 h_1 u_1 = Q_1,$$

$$\frac{\partial \mathcal{E}}{\partial h_2} = \frac{\gamma_2 u_2^2}{2} + g\gamma_2(h_1 + h_2) = R_2, \qquad \frac{\partial \mathcal{E}}{\partial u_2} = \gamma_2 h_2 u_2 = Q_2.$$

The energy flux is given by

$$D\left(h_1, h_2\right) = \frac{Q_1}{\gamma_1} R_1 + \frac{Q_2}{\gamma_2} R_2$$

$$= \frac{Q_1}{\gamma_1}\left(\frac{Q_1^2}{2\gamma_1 h_1^2} + g(\gamma_1 h_1 + \gamma_2 h_2)\right) + \frac{Q_2}{\gamma_2}\left(\frac{Q_2^2}{2\gamma_2 h_2^2} + g\gamma_2(h_1 + h_2)\right). \qquad (16)$$

The Hessian of D is :

$$D''(h_1, h_2) = \begin{pmatrix} \frac{3Q_1^3}{\gamma_1^2 h_1^4} & 0 \\ 0 & \frac{3Q_2^3}{\gamma_2^2 h_2^4} \end{pmatrix}.$$

The *extremum* point of D is given by :

$$\frac{Q_1}{\gamma_1}\left(-\frac{Q_1^2}{\gamma_1 h_1^3} + g\gamma_1\right) + \frac{Q_2}{\gamma_2} g\gamma_2 = 0,$$

$$\frac{Q_1}{\gamma_1} g\gamma_2 + \frac{Q_2}{\gamma_2}\left(-\frac{Q_2^2}{\gamma_2 h_2^3} + g\gamma_2\right) = 0.$$

Or

$$h_1^3 = \frac{Q_1^3}{g\gamma_1^2\left(Q_1 + Q_2\right)}, \qquad (17)$$

$$h_2^3 = \frac{Q_2^3}{g\gamma_2^2\left(\frac{\gamma_2}{\gamma_1} Q_1 + Q_2\right)}.$$

Two cases should be distinguished.

- the flow in both layers is in the same direction, i.e. the signs of the mass fluxes are the same. For definiteness, we suppose that the fluxes are positive, i.e. the flow is from the left to the right :

$$Q_1 > 0, \qquad Q_2 > 0. \qquad (18)$$

- The signs of the mass fluxes are opposite. One can choose, for example,

$$Q_1 > 0, \qquad Q_2 < 0. \qquad (19)$$

The case (19) is more difficult because some additional singularities will appear in (17). In the following we will consider only the case (18). In this case, the extremum point is a minimum point, and the level curves $D\left(h_1, h_2\right)$ are also closed.

At the minimum points of $M(h_1, h_2)$ and $D(h_1, h_2)$(denoted by M_M and D_M, respectively), the velocities of each layer u_1, u_2 are related to the corresponding depths h_1, h_2 by (15) and (17). One can prove that the points M_M and D_M always belong to the hyperbolicity region (the correponding characteristic polynomial (9) has three positive and one zero root at these points). The fact that one of the roots is always zero, is a very important property which is worth to be discussed in details. Consider the critical curve :

$$C(h_1, h_2) = \left(\frac{Q_1^2}{\gamma_1^2 h_1^2} - gh_1\right)\left(\frac{Q_2^2}{\gamma_2^2 h_2^2} - gh_2\right) - g^2 \frac{\gamma_2}{\gamma_1} h_1 h_2 = 0. \tag{20}$$

obtained from the characteristic polynomial (9) by taking $\lambda = 0$.

Theorem 1.*If (h_1, h_2) are extrema of $M(h_1, h_2)$ and $D(h_1, h_2)$ defined, respectivement, by (15) and (17) then $C = 0$, i.e. they belong to the critical curve.*

Proof. The minimum point of M is defined by (15) and satisfies the relations

$$\frac{Q_1^2}{\gamma_1^2 h_1^2} = g\left(h_1 + \frac{\gamma_2}{\gamma_1} h_2\right),$$

$$\frac{Q_2^2}{\gamma_2^2 h_2^2} = g(h_2 + h_1).$$

Hence

$$C = \left(\frac{Q_1^2}{\gamma_1^2 h_1^2} - gh_1\right)\left(\frac{Q_2^2}{\gamma_2^2 h_2^2} - gh_2\right) - g^2 \frac{\gamma_2}{\gamma_1} h_1 h_2$$

$$= \left(g\left(h_1 + \frac{\gamma_2}{\gamma_1} h_2\right) - gh_1\right)(g(h_2 + h_1) - gh_2) - g^2 \frac{\gamma_2}{\gamma_1} h_1 h_2 = 0.$$

The minimum point of D is defined by (17) and satisfies the relations

$$\frac{Q_1^2}{\gamma_1^2 h_1^2} = \frac{g(Q_1 + Q_2)}{Q_1} h_1,$$

$$\frac{Q_2^2}{\gamma_2^2 h_2^2} = \frac{g\left(\frac{\gamma_2}{\gamma_1} Q_1 + Q_2\right)}{Q_2} h_2.$$

Hence

$$C = \left(\frac{Q_1^2}{\gamma_1^2 h_1^2} - gh_1\right)\left(\frac{Q_2^2}{\gamma_2^2 h_2^2} - gh_2\right) - g^2 \frac{\gamma_2}{\gamma_1} h_1 h_2$$

$$= \left(\frac{g(Q_1 + Q_2)}{Q_1} h_1 - gh_1\right)\left(\frac{g\left(\frac{\gamma_2}{\gamma_1} Q_1 + Q_2\right)}{Q_2} h_2 - gh_2\right) - g^2 \frac{\gamma_2}{\gamma_1} h_1 h_2 = 0.$$

The theorem is proved.

Remark 1. *The line of equal velocities $u_1 = u_2$ which can be written as :*

$$\frac{Q_1}{\gamma_1 h_1} - \frac{Q_2}{\gamma_2 h_2} = 0$$

is always above the minimum points of $M(h_1, h_2)$ *and* $D(h_1, h_2)$*in the plane* (h_1, h_2).

Indeed, consider the minimum point of $M(h_1, h_2)$. We obtain

$$\frac{Q_1^2}{\gamma_1^2 h_1^2} = u_1^2 = g\left(h_1 + \frac{\gamma_2}{\gamma_1} h_2\right),$$

$$\frac{Q_2^2}{\gamma_2^2 h_2^2} = u_2^2 = g\left(h_2 + h_1\right).$$

Hence

$$\frac{Q_1^2}{\gamma_1^2 h_1^2} - \frac{Q_2^2}{\gamma_2^2 h_2^2} = u_1^2 - u_2^2 = g h_2 \left(\frac{\gamma_2}{\gamma_1} - 1\right) < 0,$$

i.e. $u_1 - u_2 < 0$ at the minimum point of $M(h_1, h_2)$.

Consider now the minimum point of $D(h_1, h_2)$. We get

$$\frac{Q_1^3}{\gamma_1^3 h_1^3} = \frac{g\left(Q_1 + Q_2\right)}{\gamma_1},$$

$$\frac{Q_2^3}{\gamma_2^3 h_2^3} = \frac{g\left(\frac{\gamma_2}{\gamma_1} Q_1 + Q_2\right)}{\gamma_2}.$$

Hence

$$\frac{Q_1^3}{\gamma_1^3 h_1^3} - \frac{Q_2^3}{\gamma_2^3 h_2^3} = u_1^3 - u_2^3 = \frac{g\left(Q_1 + Q_2\right)}{\gamma_1} - \frac{g\left(\frac{\gamma_2}{\gamma_1} Q_1 + Q_2\right)}{\gamma_2} = g\frac{Q_2}{\gamma_2}\left(\frac{\gamma_2}{\gamma_1} - 1\right) < 0,$$

i.e. $u_1 - u_2 < 0$ at the minimum point of $D(h_1, h_2)$. One can also show that the point D_M is situated above the point M_M. In Figure 2, we show a typical relative position of the curves $\mathcal{C}$ =0, $M = const$, $D = const$, $u_1 = u_2$ and the minimum points D_M and M_M at the plane (h_1, h_2).

Theorem 2. *The curves* $M(h_1, h_2) = const$ *and* $D(h_1, h_2) = const$ *are tangent at a point of the plane* (h_1, h_2)*if and only if at this point* $u_1 = u_2$ *or* $\mathcal{C}$ =0.

Proof. Consider the curves $M(h_1, h_2) = const$ and $D(h_1, h_2) = const$ defined by (14), (16)

$$\begin{aligned} M(h_1, h_2) &= \frac{Q_1^2}{\gamma_1 h_1} + \frac{Q_2^2}{\gamma_2 h_2} + g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2, \\ D(h_1, h_2) &= \frac{Q_1}{\gamma_1}\left(\frac{Q_1^2}{2\gamma_1 h_1^2} + g(\gamma_1 h_1 + \gamma_2 h_2)\right) + \frac{Q_2}{\gamma_2}\left(\frac{Q_2^2}{2\gamma_2 h_2^2} + g\gamma_2(h_1 + h_2)\right). \end{aligned}$$

The curves are tangent if and only if

$$0 = \frac{\partial M}{\partial h_1}\frac{\partial D}{\partial h_2} - \frac{\partial D}{\partial h_1}\frac{\partial M}{\partial h_2} = \left(-\frac{Q_1^2}{\gamma_1 h_1^2} + g\gamma_1 h_1 + g\gamma_2 h_2\right)\left(g\left(Q_2 + Q_1\frac{\gamma_2}{\gamma_1}\right) - \frac{Q_2^3}{\gamma_2^2 h_2^3}\right) -$$

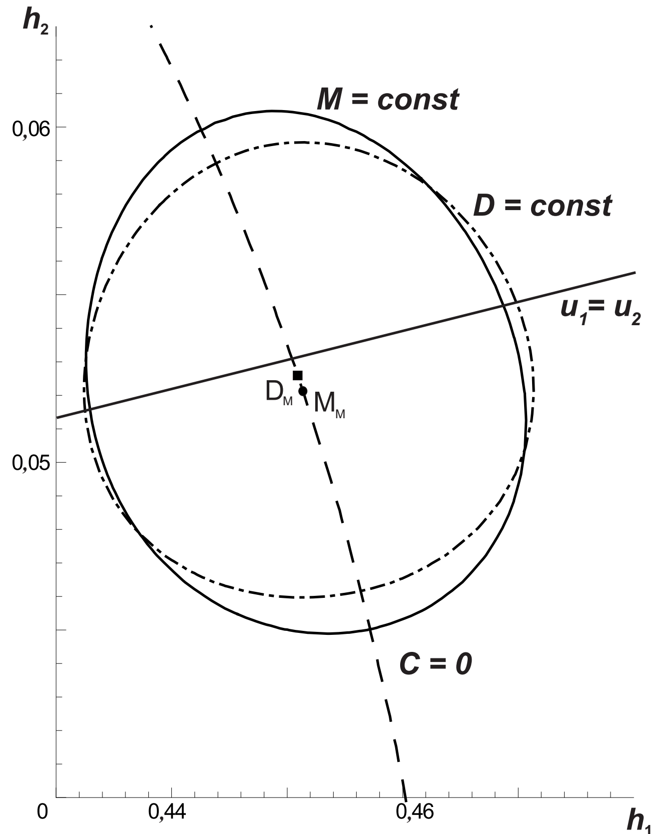

Figure 2: A typical behaviour of the curves $C = 0$, $M = const$, $D = const$, and $u_1 = u_2$ is shown. The minimum points D_M and M_M are always below the line $u_1 = u_2$, and M_M is always below D_M.

$$-\left(-\frac{Q_2^2}{\gamma_2 h_2^2}+g\gamma_2 h_1+g\gamma_2 h_2\right)\left(g\left(Q_2+Q_1\right)-\frac{Q_1^3}{\gamma_1^2 h_1^3}\right)=$$

$$=-\frac{\left(Q_2\gamma_1 h_1-Q_1\gamma_2 h_2\right)}{h_1^3 h_2^3\gamma_1^2\gamma_2^2}\left(Q_1^2\left(-Q_2^2+gh_2^3\gamma_2^2\right)+gh_1^3\gamma_1\left(Q_2^2\gamma_1+gh_2^3\gamma_2^2\left(\gamma_2-\gamma_1\right)\right)\right)=$$

$$=g^2\left(Q_2\gamma_1 h_1-Q_1\gamma_2 h_2\right)\left(\left(\frac{Q_2^2}{gh_2^3\gamma_2^2}-1\right)\left(\frac{Q_1^2}{gh_1^3\gamma_1^2}-1\right)-\frac{\gamma_2}{\gamma_1}\right)$$

Hence, the Jacobian matrix is degenerate if and only if

$$Q_2\gamma_1 h_1-Q_1\gamma_2 h_2=0$$

or

$$\left(\frac{Q_2^2}{gh_2^3\gamma_2^2}-1\right)\left(\frac{Q_1^2}{gh_1^3\gamma_1^2}-1\right)-\frac{\gamma_2}{\gamma_1}=0.$$

This is equivalent to relations $u_2=u_1$ or $C=0$.

3 Supercritical-subcritical transition

Consider a discontinuous piecewise constant stationary solulution of (1) - (4)

$$\left(h_1\left(x\right),h_2\left(x\right),u_1\left(x\right),u_2\left(x\right)\right)^T=\begin{cases}\left(h_{10},h_{20},u_{10},u_{20}\right)^T, & \text{if } x<0\\ \left(h_1,h_2,u_1,u_2\right)^T, & \text{if } x>0\end{cases}$$

For convenience, we will use the same notations for the right state constant variables as for the basic variables (without additional indices). The critical curve $C\left(h_1,h_2\right)=0$ is defined by (20). The flow is *supercritical* if $C>0$ and *subcritical* if $C<0$. Supppose that the flow satisfies $Q_1>0$, $Q_2>0$ and the state "0" at the left is supercritical. The constant states satisfy the relations (10), (11) and (12). An additional relation is needed satisfying the natural dissipation inequality (very often called in the context of hyperbolic equations "entropy inequality") :

$$\left[D\left(h_1,h_2\right)\right]=\left[\frac{Q_1}{\gamma_1}R_1+\frac{Q_2}{\gamma_2}R_2\right]$$

$$=\left[\frac{Q_1}{\gamma_1}\left(\frac{Q_1^2}{2\gamma_1 h_1^2}+g(\gamma_1 h_1+\gamma_2 h_2)\right)+\frac{Q_2}{\gamma_2}\left(\frac{Q_2^2}{2\gamma_2 h_2^2}+g\gamma_2(h_1+h_2)\right)\right]\leq 0.$$

where

$$[D]=D\left(h_1,h_2\right)-D\left(h_{10},h_{20}\right)$$

Since the curve $D\left(h_1,h_2\right)=D\left(h_{10},h_{20}\right)$ is closed, the "entropy inequality" means that the subcritical state should be inside this closed curve. In particular, the entropy inequality must imply that the state at the right is subcritical.

For given positive Q_1, Q_2, we choose $\left(h_{10},h_{20}\right)$ such that the whole closed curve $M\left(h_1,h_2\right)=M\left(h_{10},h_{20}\right)$ belongs to the hyperbolicity domain (see Figure

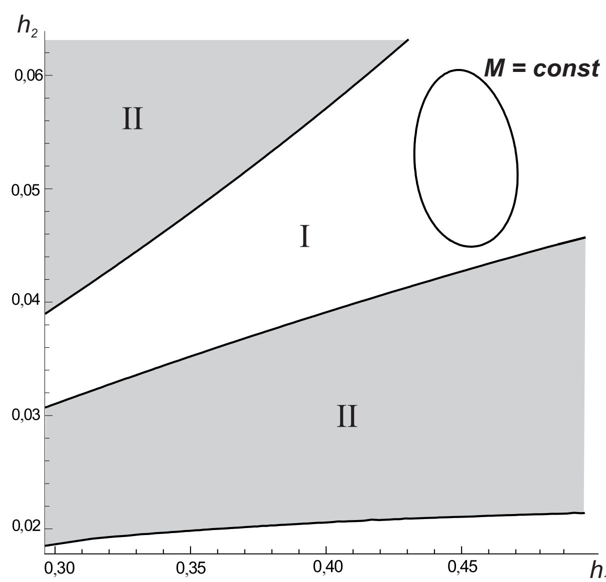

Figure 3: The hyperbolicity domain is non-colored, while the ellipticity domain is in gray (we have there two real and two complex conjugate eigenvalues of the polynomial (9)). The curve $M = const$ is chosen to be entirely in the hyperbolicity region.

3). This allows us to avoid additional mathematical difficulties relating to the interpretation of the solution in the elliptic domain.

The curve $M(h_1, h_2) = M(h_{10}, h_{20})$ can be parametrized, if we introduce the polar coordinates (r, θ) :

$$h_1 = r(\theta)\cos\theta, \qquad h_2 = r(\theta)\sin\theta, \quad \theta \in \left(\theta^-, \theta^+\right).$$

Here $r(\theta)$ is determined from the third order polynomial coming from the relation

$$\frac{Q_1^2}{\gamma_1 h_1} + \frac{Q_2^2}{\gamma_2 h_2} + g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2 = \mathcal{M} = const.$$

It has the form

$$r^3 g\left(\gamma_1 (\cos\theta)^2/2 + \gamma_2\cos\theta\sin\theta + \gamma_2(\sin\theta)^2/2\right) - Mr + \frac{Q_1^2}{\gamma_1\cos\theta} + \frac{Q_2^2}{\gamma_2\sin\theta} = 0. \tag{21}$$

One of the roots of this polynomial is always negative (hence, non-physical), the two other roots are real and positive : $r_1(\theta) < r_2(\theta)$, for $\theta \in (\theta^-, \theta^+)$. The union of these two brunches form a smooth closed curve $M = const$ in the plane (r, θ) : $r_1(\theta^-) = r_2(\theta^-)$, $r_1(\theta^+) = r_2(\theta^+)$ (see Figure 4). Consider now the stationary

non-conservative momentum equations (1), (2) for each layer :

$$\gamma_1 h_1 u_1 u_{1x} + g h_1(\gamma_1 h_1 + \gamma_2 h_2)_x = 0,$$

$$\gamma_2 h_2 u_2 u_{2x} + g h_2 \gamma_2 (h_1 + h_2)_x = 0.$$

They become conservative along the curve $M = const$. Indeed, the non-conservative product $h_1(h_2)_x$ is now :

$$h_1(h_2)_x = r(\theta)\cos\theta\left(\frac{dr(\theta)}{d\theta}\theta_x\sin\theta + r(\theta)\theta_x\cos\theta\right) = \left(\frac{r^2(\theta)}{2}\sin\theta\cos\theta + \frac{1}{2}\int_{\theta_*}^{\theta} r^2(\theta)d\theta\right)_x =$$

$$= \left(\frac{h_1 h_2}{2} + \frac{1}{2}\int_{\theta_*}^{\theta} r^2(\theta)d\theta\right)_x.$$

Analogously,

$$h_2(h_1)_x = r(\theta)\sin\theta\left(\frac{dr(\theta)}{d\theta}\theta_x\cos\theta - r(\theta)\theta_x\sin\theta\right) = \left(\frac{r^2(\theta)}{2}\sin\theta\cos\theta - \frac{1}{2}\int_{\theta_*}^{\theta} r^2(\theta)d\theta\right)_x =$$

$$= \left(\frac{h_1 h_2}{2} - \frac{1}{2}\int_{\theta_*}^{\theta} r^2(\theta)d\theta\right)_x.$$

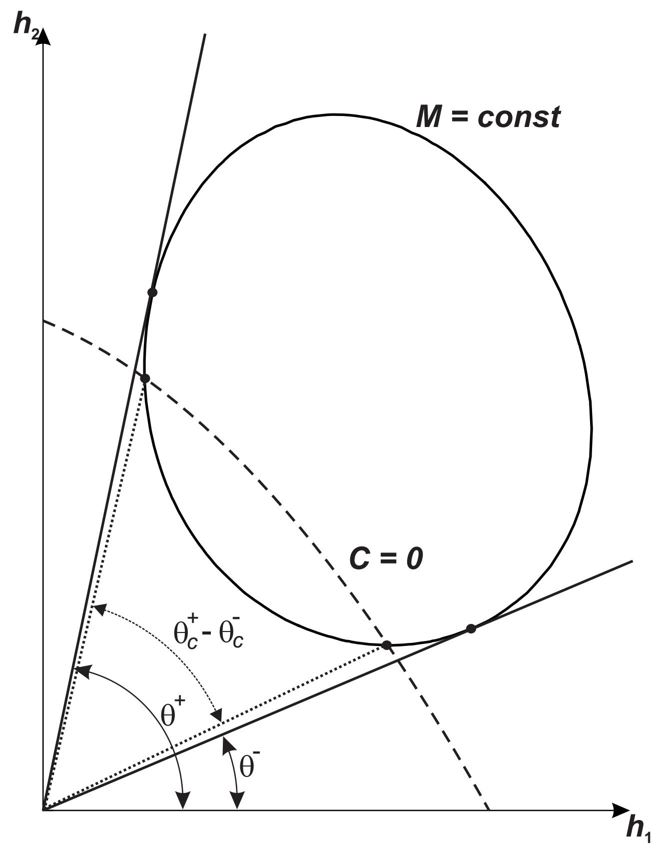

Figure 4: The momentum curve $M = const$ can be parametrized in polar coordinates in the plane (h_1, h_2).

Here θ_* is any constant belonging to (θ^-, θ^+). Finally, one of the following conserved quantities can be chosen:

$$\frac{Q_1^2}{\gamma_1 h_1} + g\left(\frac{\gamma_1 h_1^2}{2} + \gamma_2\left(\frac{h_1 h_2}{2} + \frac{1}{2}\int_{\theta_*}^{\theta} r^2(\theta)d\theta\right)\right) = \frac{Q_1^2}{\gamma_1 h_{10}} + g\left(\frac{\gamma_1 h_{10}^2}{2} + \gamma_2\left(\frac{h_{10} h_{20}}{2} + \frac{1}{2}\int_{\theta_*}^{\theta_0} r^2(\theta)d\theta\right)\right) = M_1,$$

$$\frac{Q_2^2}{\gamma_2 h_2} + g\left(\frac{\gamma_2 h_2^2}{2} + \gamma_2\left(\frac{h_1 h_2}{2} - \frac{1}{2}\int_{\theta_*}^{\theta} r^2(\theta)d\theta\right)\right) = \frac{Q_2^2}{\gamma_2 h_{20}} + g\left(\frac{\gamma_2 h_{20}^2}{2} + \gamma_2\left(\frac{h_{10} h_{20}}{2} - \frac{1}{2}\int_{\theta_*}^{\theta_0} r^2(\theta)d\theta\right)\right) = M_2.$$

This system is compatible with the total momentum. Indeed, summing them we obtain

$$M_1 + M_2 = M.$$

We can choose the conservation of the local momentum M_1 (or M_2) as an additional closure relation. We have now to check if the corresponding state is in the subcritical region, and that the "entropy inequality" is satisfied. For definiteness, we choose the local momentum equation of the first layer. Obviously, this relation does not depend on the choice of θ_*. Finally, for a given supercritical state h_{10}, h_{20} we have to find a subcritical state h_1, h_2 from the following system of equations :

$$\frac{Q_1^2}{\gamma_1 h_1} + \frac{g\gamma_1 h_1^2}{2} + \frac{g\gamma_2 h_1 h_2}{2} + \frac{g\gamma_2}{2}\int_{\theta_0}^{\theta} r^2(\theta)d\theta = \frac{Q_1^2}{\gamma_1 h_{10}} + \frac{g\gamma_1 h_{10}^2}{2} + \frac{g\gamma_2 h_{10} h_{20}}{2}, \quad (22)$$

$$\frac{Q_1^2}{\gamma_1 h_1} + \frac{Q_2^2}{\gamma_2 h_2} + g\gamma_1 h_1^2/2 + g\gamma_2 h_1 h_2 + g\gamma_2 h_2^2/2 = \frac{Q_1^2}{\gamma_1 h_{10}} + \frac{Q_2^2}{\gamma_2 h_{20}} + g\gamma_1 h_{10}^2/2 + g\gamma_2 h_{10} h_{20} + g\gamma_2 h_{20}^2/2. \quad (23)$$

Here

$$\theta_0 = arctan\left(\frac{h_{20}}{h_{10}}\right), \quad \theta = arctan\left(\frac{h_2}{h_1}\right),$$

and $r(\theta)$ is given by (21).

4 Solution algorithm to find the subcritical state

The following particular parameters were chosen to calculate the constant $M(h_{10}, h_{20})$:

$$h_{10} = 0.45, \quad h_{20} = 0.045, \quad Q_1 = 10, \quad Q_2 = 1.1, \quad \gamma_1 = 10, \quad \gamma_2 = 9.5.$$

They determine the curve $M(h_1, h_2) = M(h_{10}, h_{20})$ entirely belonging to the hyperbolicity region. In the following, (h_{10}, h_{20}) will mean any generic point at this fixed curve, and not only these chosen particular values. It is more simpler to see the solution structure in the plane (r, θ). The curve $C = 0$ cuts the curve $M = const$ at points corresponding to angles $\theta_c^- < \theta_c^+$ such that

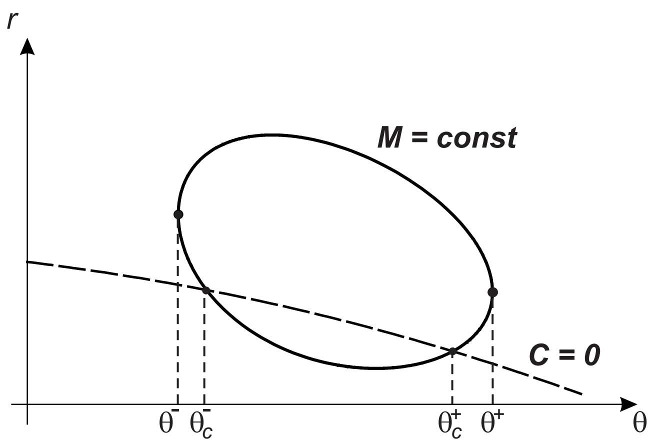

Figure 5: A typical behaviour of the curves $C = 0$, $M = const$ in (r, θ) - plane is shown.

$\theta^- < \theta_c^- < \theta_c^+ < \theta^+$ (see Figure 5). First, it was numerically shown that for any initial subcritical state belonging to the curve $r_1(\theta_0)$, $\theta_c^- < \theta_0 < \theta_c^+$, there is no non-trivial solution of the system (22), (23) with θ belonging to the brunch $r_1(\theta)$, $\theta^- < \theta < \theta^+$. Hence, the solution may be only on the brunch $r_2(\theta)$, $\theta^- < \theta < \theta^+$. The second step was to present

$$\int\limits_{\theta_0}^{\theta} r^2(\theta)d\theta = \int\limits_{\theta_0}^{\theta^-} r_1^2(\theta)d\theta + \int\limits_{\theta^-}^{\theta} r_2^2(\theta)d\theta$$

and look for a non-trivial solution θ with $r = r_2(\theta)$. It was found that this value of θ was unique and always satisfied the "entropy inequality". A typical picture is shown in (Figure 6 where S is an initial supercritical state, and J is the corresponding subcritical state. The following property was also established : if the initial supercritical data vary monotonically with respect to the angle θ_0 (they are denoted by points enumerated from 1 to 8 in Figure 7), the same monotonicity is observed for the subcritical states (they are denoted by squares enumerated from 1 to 8) (see Figure 7).

5 Comparison with other approach based on the Bernoulli integrals

An alternative approach consists in choosing the conservation law for the velocity difference as a closing jump relation :

$$(u_1 - u_2)_t + \left(\frac{u_1^2}{2} - \frac{u_2^2}{2} + gh_2\left(\frac{\gamma_2}{\gamma_1} - 1\right)\right)_x = 0.$$

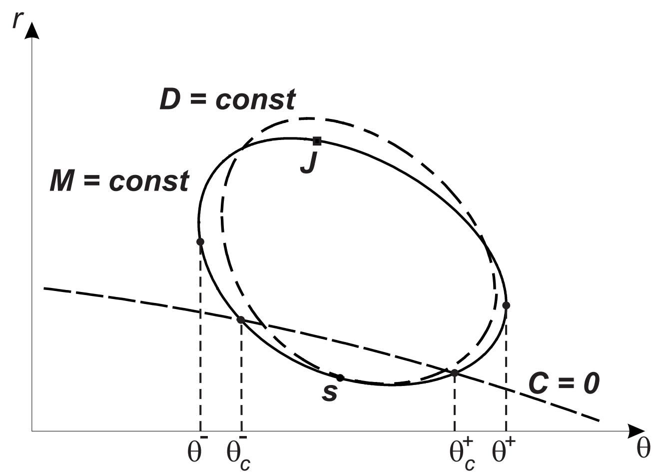

Figure 6: The supercritical state S and the subcritical state J belong to the same closed solid curve $M = const$. The state J satisfies the "entropy inequality" : it is inside the dashed closed curve $D = const$ passing by S.

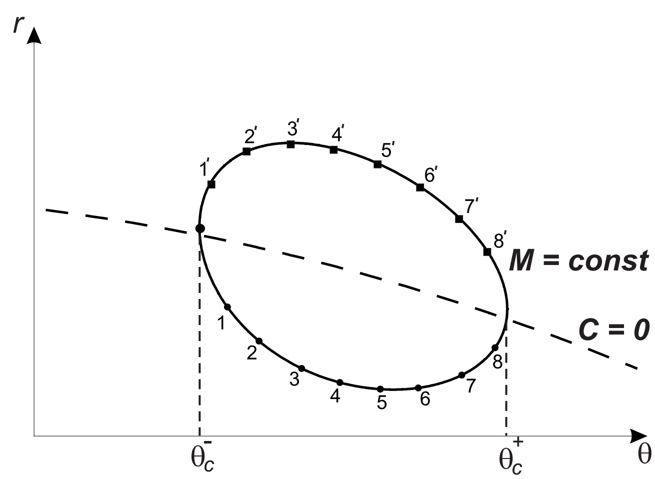

Figure 7: 8 initial states were uniformly distributed at the supercritical part of the curve $M = const$ (shown by points). The corresponding subcritical states are shown by squares.

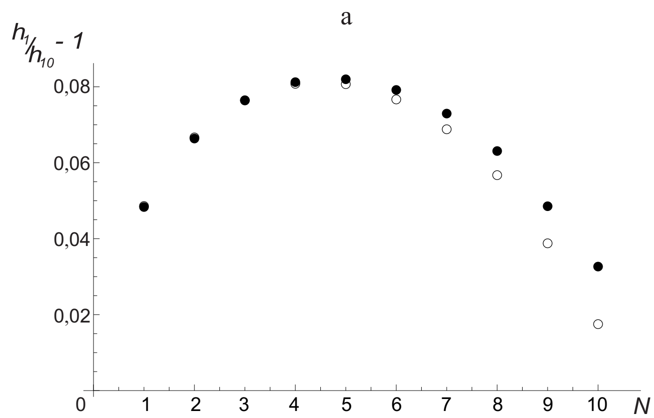

Figure 8: The deviation of the internal jump for an array of ten points is shown for a new approach (by dots) and the Bernoulli approach (by empty dots).

For stationary hydraulic jumps, it corresponds to the conservation of the quantity :

$$\frac{u_1^2}{2} - \frac{u_2^2}{2} + gh_2\left(\frac{\gamma_2}{\gamma_1} - 1\right) = const$$

It can be shown that in both approaches, the "entropy inequality" is satisfied. For ten points distributed uniformly at the subcritical part of the curve $M = const$ and enumerated in increasing order with respect to θ, the results for the corresponding subcritical states are shown in Figures 8 and 9. For both, internal and surface jumps, the amplitudes of jumps obtained by a new method are larger than those obtained by using the difference of the Bernoulli integrals. One can also numerically prove that in the new approach the following inequality is always satisfied :

$$\frac{h_1}{h_{10}} > \frac{h_1 + h_2}{h_{10} + h_{20}} \tag{24}$$

i.e. the amplitude of the internal jump is always larger than the amplitude of the jump corresponding to surface waves. When the Bernoulli approach is used, the inequality (24) is not always satisfied (Figure 10). The inequality (24) corroborates a well-known fact concerning two-layer flows (Armi, 1986) : "two-layer flows with a free surface and small non-dimensional-density difference behave exactly as bounded two-layer flows with the upper boundary level".

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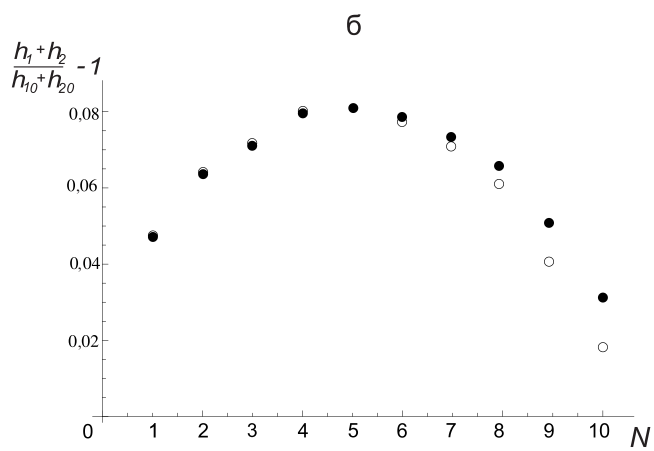

Figure 9: The deviation of the surface jump for an array of ten points is shown for a new approach (by squares) and the Bernoulli approach (by empty squares).

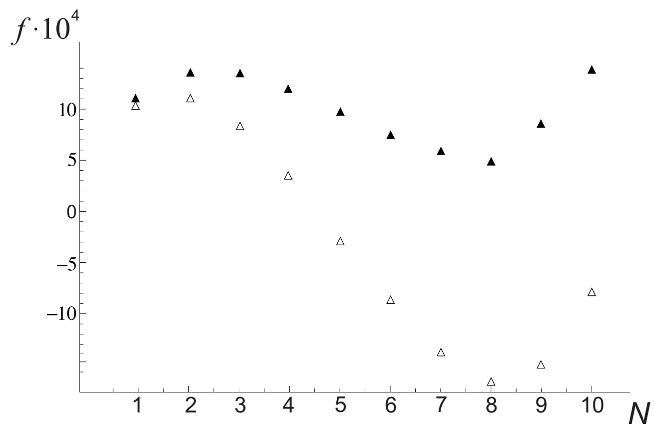

Figure 10: For the new approach, the amplitude of the internal jump is always larger than the amplitude of the jump corresponding to surface waves (shown by triangles), while for the Bernoulli approach it is not always the case (shown by empty triangles).

6 References

Abgrall, R. and Karni, S. 2009, Two-layer shallow water systems : a relaxation approach, SIAM J. of Scientific Computing, **31** (3), 1603-1627.

Armi, L. 1986 The hydraulics of two flowing layers with different densities, J. Fluid Mechanics, **163**, pp. 27-58.

Baines, P. G. 1995 Topographic effects in stratified flows, Cambridge University Press, Cambridge.

Barros, B. 2006 Conservation laws for one-dimensional shallow water models for one and two-layer flows, Math. Models and Methods in Applied Sciences. **16**, 119-137.

Barros, R., Gavrilyuk, S. and Teshukov, V. 2007 Dispersive Nonlinear Waves in Two-Layer Flows with Free Surface. I. Model Derivation and General Properties, Studies in Applied Mathematics, 119: 191-211.

Barros, R. and Gavrilyuk, S. 2007 Dispersive Nonlinear Waves in Two-Layer Flows with Free Surface. II. Large Amplitude Solitary Waves Embedded into the Continuous Spectrum, Studies in Applied Mathematics, 119: 213-251.

Bridges, T.J. and Donaldson, N. M. 2009 Criticality manifolds and their role in the generation of solitary waves for two-layer flow with a free surface, European Journal of Mechanics B/Fluids 28, 117-126.

El, G. A., Grimshaw, R. H. J. and Smyth, N. F. 2006, Unsteady undular bores in fully nonlinear shallow-water theory, Physics of Fluids, 18:027104 (17 pages).

Le Metayer, O., Gavrilyuk, S. and Hank, S. 2010 A numerical scheme for the Green-Naghdi model, J. Comp. Physics, **229**, 2034–2045.

Liapidevskii and Teshukov (2000) Mathematical models of propagation of long waves in non-homogeneous fluids. Edited by Siberian Division of the Russian Academy of Sciences, Novosibirsk.

Liska, R., Margolin, L. and Wendroff, B. 1995 Nonhydrostatic two-layer models of incompressible flow, Computers Math. Appl. 29 : 25-37.

Mandli, K. T. 2011 Finite volume methods for the multilayer shallow water equations with applications to storm surges, PhD thesis, University of Washington.

Montgomery, P.J. and Moodie, 2001 On the number of conserved quantities for the two-layer shallow water equations, Studies in Applied Mathematics **106,** 229-259.

Ostapenko, V.V. 2001 Stable shock waves in two layer "shallow water" , J. Applied Mathematics and Mechanics, 65 (1), 89-108.

Ovsyannikov, L. V. 1979, Two-layer "shallow water" model, J. Applied Mechanics and Technical Physics, N 2, 3-14.

Ovsyannikov, L. V. *et al.* 1985 Nonlinear problems in the theory of surface and internal waves, Novosibirsk, Nauka (in russian).

Stoker, J. J. 1957, Water Waves, Interscience, New York.